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ON THE POSITIVE PELLIAN EQUATION $y^2=6x^2+10$

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ABSTRACT

The binary quadratic equation represented by the positive Pellian $y^2=6x^2+10$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola and special pythagorean triangle.

KEYWORDS: *Binary Quadratic, Hyperbola, Parabola, Pell Equation, Integral Solutions*

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INTRODUCTION

A binary quadratic equation of the form $y^2=Dx^2+1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions. When D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, a hyperbola given by $y^2=6x^2+10$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is

$$y^2=6x^2+10 \tag{1}$$

whose smallest positive integer solution (x_0,y_0) is

$$x_0=1,\ y_0=4$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2=6x^2+1$$

whose smallest positive integer solution is

$$\tilde{x}_0=2,\ \tilde{y}_0=5 \tag{2}$$

Whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{6}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

Where

$$f_n = \left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1},$$

$$g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}, \quad n = -1, 0, 1 \ldots$$

To obtain the sequence of solutions of (1), we employ the lemma known as Brahmagupta lemma stated as follows:

If (x_0, y_0) and (x_1, y_1) represent the solutions of the pell equations $y^2 = Dx^2 + k_1$ and $y^2 = Dx^2 + k_2$ $(D > 0 \ \textit{and square free})$ respectively, then $(x_0 y_1 + y_0 x_1,\ y_0 y_1 + D x_0 x_1)$ represents the solution of the pell equation $y^2 = Dx^2 + k_1 k_2$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{2}{\sqrt{6}} g_n$$

$$y_{n+1} = 2 f_n + \frac{3}{\sqrt{6}} g_n$$

$$\Rightarrow \quad 2\sqrt{6}\, x_{n+1} = \sqrt{6} f_n + 4 g_n \tag{3}$$

$$\sqrt{6}\, y_{n+1} = 2\sqrt{6} f_n + 3 g_n \tag{4}$$

Replacing n by $n+1$ in (3), we get

$$2\sqrt{6}\, x_{n+2} = \sqrt{6} f_{n+1} + 4 g_{n+1}$$

$$2\sqrt{6}\, x_{n+2} = \sqrt{6}\left(5 f_n + 2\sqrt{6} g_n\right) + 4\left(5 g_n + 2\sqrt{6} f_n\right)$$

$$2\sqrt{6}\, x_{n+2} = 13\sqrt{6} f_n + 32 g_n \tag{5}$$

Replacing n by $n+1$ in (5), we get

$$2\sqrt{6}\, x_{n+3} = 13\sqrt{6} f_{n+1} + 32 g_{n+1}$$

$$2\sqrt{6}x_{n+3}=13\sqrt{6}\left(5f_n+2\sqrt{6}g_n\right)+32\left(5g_n+2\sqrt{6}f_n\right)$$

$$2\sqrt{6}x_{n+3}=129\sqrt{6}f_n+316g_n \quad (6)$$

Replacing n by $n+1$ in (4), we get

$$\sqrt{6}y_{n+2}=2\sqrt{6}f_{n+1}+3g_{n+1}$$

$$=2\sqrt{6}\left(5f_n+2\sqrt{6}g_n\right)+3\left(5g_n+2\sqrt{6}f_n\right)$$

$$\sqrt{6}y_{n+2}=16\sqrt{6}f_n+39g_n \quad (7)$$

Replacing n by $n+1$ in (7), we get

$$\sqrt{6}y_{n+3}=16\sqrt{6}f_{n+1}+39g_{n+1}$$

$$=16\sqrt{6}\left(5f_n+2\sqrt{6}g_n\right)+39\left(5g_n+2\sqrt{6}f_n\right)$$

$$\sqrt{6}y_{n+3}=158\sqrt{6}f_n+387g_n \quad (8)$$

These are representing the non-zero distinct integer solutions of (1)

A few numerical examples are given in the following Table 1

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	4
0	13	32
1	129	316
2	1277	3128
3	12641	30964
4	125133	306512

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively,

$$x_{n+3}-10x_{n+2}+x_{n+1}=0,\ n=-1,0,1....$$

$$y_{n+3}-10y_{n+2}+y_{n+1}=0,\ n=-1,0,1....$$

A few interesting relations among the solutions are given below:

x_{n+1} **is always odd,** y_{n+1} **is always even and** $y_{n+1}\equiv 0\pmod 4$.

Relations among the Solutions

- $5x_{n+2} - 10y_{n+1} - 25x_{n+1} = 0$
- $25x_{n+2} - 5x_{n+1} - 10y_{n+2} = 0$
- $245x_{n+2} - 25x_{n+1} - 10y_{n+3} = 0$
- $-25y_{n+1} - 60x_{n+1} + 5y_{n+2} = 0$
- $-5y_{n+1} - 60x_{n+2} + 25y_{n+2} = 0$
- $-25y_{n+1} - 60x_{n+3} + 245y_{n+2} = 0$

Nasty Numbers

Solving (3) and (5), we get

$$f_n = \frac{2}{5}(x_{n+2} - 8x_{n+1}) \quad (9)$$

$$g_n = \frac{\sqrt{6}}{10}(13x_{n+1} - x_{n+2}) \quad (10)$$

Replacing n by $2n+1$ in (9), we have

$$f_{2n+1} = \frac{2}{5}(x_{2n+3} - 8x_{2n+2})$$

Note that,

$$f_{2n+1} + 2 = f_n^2$$

Now,

$$6\left[\frac{2}{5}(x_{2n+3} - 8x_{2n+2}) + 2\right] = 6f_n^2$$

$N_1 = \left[\frac{12}{5}(x_{2n+3} - 8x_{2n+2}) + 12\right]$ is a Nasty number.

The other choices of Nasty numbers are presented below

- $N_2 = \left[\frac{3}{5}(13y_{2n+2} - y_{2n+3}) + 12\right]$
- $N_3 = \left[\frac{12}{25}(2y_{2n+3} - 39x_{2n+2}) + 12\right]$

- $N_4 = \left[\frac{6}{25}\left(32y_{2n+2} - 6x_{2n+3}\right) + 12\right]$

Cubical Integers

Replacing n by $3n+2$ in (9), we have

$$f_{3n+2} = \frac{2}{5}\left(x_{3n+4} - 8x_{3n+3}\right)$$

Now,

$$f_{3n+2} = f_n^3 - 3f_n$$

$$f_{3n+2} + 3f_n = f_n^3$$

$$\Rightarrow f_n^3 = \frac{2}{5}\left(x_{3n+4} - 8x_{3n+4}\right) + \frac{6}{5}\left(\left(x_{n+2} - 8x_{n+1}\right)\right)$$

$$\Rightarrow C_1 = \frac{2}{5}\left(x_{3n+4} - 8x_{3n+4}\right) + \frac{6}{5}\left(\left(x_{n+2} - 8x_{n+1}\right)\right)$$ is a Cubical integer.

The other choices of Cubical integers are presented below:

- $C_2 = \frac{1}{10}\left(13y_{3n+3} - y_{3n+4}\right) + \frac{3}{10}\left(13\left(y_{n+1} - y_{n+2}\right)\right)$
- $C_3 = \frac{2}{25}\left(2y_{3n+4} - 39x_{3n+3}\right) + \frac{6}{25}\left(\left(2y_{n+2} - 39x_{n+1}\right)\right)$
- $C_4 = \frac{1}{25}\left(32y_{3n+3} - 6x_{3n+4}\right) + \frac{3}{25}\left(\left(32y_{n+1} - 6x_{n+2}\right)\right)$

Bi-Quadratic Integers

Replacing n by $4n+3$ in (9), we have

$$f_{4n+3} = \frac{2}{5}\left(x_{4n+5} - 8x_{4n+4}\right)$$

Now, $f_{4n+3} + 4f_n^2 - 2 = f_n^4$

$$\Rightarrow f_n^4 = \frac{2}{5}\left(x_{4n+5} - 8x_{4n+4}\right) + 4\left[\frac{2}{5}\left(x_{2n+3} - 8x_{2n+2}\right) + 2\right] - 2$$

$$\mathbf{B_1} = \frac{2}{5}\left(x_{4n+5} - 8x_{4n+4}\right) + \left[\frac{8}{5}\left(x_{2n+3} - 8x_{2n+2}\right)\right] + 6$$ is a Bi-quadratic integer.

The other choices of Bi-quadratic integers are presented below:

- $\mathbf{B}_2 = \frac{1}{10}(13y_{4n+4} - y_{4n+5}) + \left[\frac{4}{10}(13y_{2n+2} - y_{2n+3})\right] + 6$
- $\mathbf{B}_3 = \frac{2}{25}(2y_{4n+5} - 39x_{4n+4}) + \left[\frac{8}{25}(2y_{2n+3} - 39x_{2n+2})\right] + 6$
- $\mathbf{B}_4 = \frac{1}{25}(32y_{4n+4} - 6x_{4n+5}) + \left[\frac{4}{25}(32y_{2n+2} - 6x_{2n+3})\right] + 6$

REMARKABLE OBSERVATIONS

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented below:

Solving (4) and (7), we get

$$f_n = \frac{1}{10}X \quad (11)$$

$$g_n = \frac{\sqrt{6}}{15}Y \quad (12)$$

where

$$X = 13y_{n+1} - y_{n+2}$$

$$Y = y_{n+2} - 8y_{n+1}$$

We know that, $f_n^2 - g_n^2 = 4$ (13)

Substituting (11) and (12) in (13), we have

$$\left[\frac{X}{10}\right]^2 - \left[\frac{\sqrt{6}}{15}Y\right]^2 = 4$$

$$\frac{1}{100}X^2 - \frac{6}{225}Y^2 = 4$$

$\Rightarrow 9X^2 - 24Y^2 = 3600$ which represents the Hyperbola.

The other choices of hyperbolas are presented in the Table: 2 below

Table 2: Hyperbolas

S.No	Hyperbolas	(X, Y)
1	$16X^2-6Y^2=400$	$(x_{n+2}-8x_{n+1}, 13x_{n+1}-x_{n+2})$
2	$4X^2-6Y^2=2500$	$(2y_{n+2}-39x_{n+1}, 32x_{n+1}-y_{n+2})$
3	$X^2-6Y^2=2500$	$(4y_{n+1}-12x_{n+2}, 4x_{n+2}-13y_{n+1})$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented below:

Solving (3) and (5), we get

$$f_n=\frac{2}{5}X \quad g_n=\frac{\sqrt{6}}{10}Y \tag{14}$$

where

$$X=(x_{n+2}-8x_{n+1})$$

$Y=(13x_{n+1}-x_{n+2})$ Replacing n by $2n+1$ in (9), we have

$$f_{2n+1}=\frac{2}{5}(x_{2n+3}-8x_{2n+2})$$

Note that

$$f_{2n+1}+2=f_n^2$$

$$\therefore \quad f_n^2=\frac{2}{5}(x_{2n+3}-8x_{2n+2})+2$$

$$\therefore \quad f_n^2=\frac{2}{5}X+2 \tag{15}$$

$$g_n^{\ 2}=\frac{6}{100}Y^2 \tag{16}$$

Substituting (15) and (16) in (13), we have

$$\frac{2}{5}X-\frac{6}{100}Y^2=2$$

$\Rightarrow 40X-6Y^2=200$ which represents a Parabola.

The other choice of parabolas is presented in the Table: 3 below:

Table 3: Parabolas

S. No	Parabolas	(X, Y)
1	$45X - 12Y^2 = 900$	$(13y_{2n+2} - y_{2n+3},\ y_{n+2} - 8y_{n+1})$
2	$18X - 6Y^2 = 900$	$(2y_{2n+3} - 39x_{2n+2},\ 32x_{n+1} - y_{n+2})$
3	$25X - 6Y^2 = 1250$	$(32y_{2n+2} - 6x_{2n+3},\ 4x_{n+2} - 13y_{n+1})$

Generation of the Pythagorean Triangle

Let p, q be the non-zero distinct integers such that $p = x_{n+1} + y_{n+1}$, $q = x_{n+1}$

Note that $p > q > 0$. Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$

$$X = 2pq\ ,\ Y = p^2 - q^2\ ,\ Z = p^2 + q^2, p > q > 0$$

Then

Z-X = $(p-q)^2$ Z-Y=$2q^2$

Let (Z-X)= 3(Z-Y) +10

$(p-q)^2 = 6q^2 + 10$

$6q^2$ =Z-X-10 (1)

Z-X=$6q^2$ +10

$Y^2 = 6X^2 + 10$:

Where

(p-q)=Y X=q

Let A,P represent the area and perimeter of T

Then

2A/P = 2(pq(p2-q2)/2p(p+q) =q(p-q)

4A/P =2pq-$2q^2$

$6q^2$ =6pq-12A/P =3(X-4A/P) (2) from (1) and (2)

So the following interesting relations are observed.

- $3Y - X - 2Z = 10$
- $Z - 4X + \frac{12A}{P} = 10$
- $3\left(X - \frac{4A}{P}\right)$ is a Nasty number

- $\frac{2A}{P} = x_{n+1} y_{n+1}$

CONCLUSIONS

In this paper we presented infinitely many integer solutions to the hyperbola represented by the positive pell equation $y^2 = 6x^2 + 10$ along with suitable relations between the solutions, Since Quadratic Diophantine equations are infinite, one may attempt to determine integer solutions of other equations of degree 2 as well as higher degree with suitable properties.

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